Symmetry, Yang-Baxter Equation, Quantum groups, and Link Invariants

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1. Introduction.

Symmetry is of central importance in current mathematics, physics, chemistry, and many other areas of inquiry. Indeed in well-known journals like Letters in Mathematical Physics and Journal of Mathematical Physics one finds that, respectively, over 50% and some 30% of the articles deal with one or another aspect of symmetry (and groups and representations of groups) (in 1992).

Most people are familiar with the intuitive idea of symmetry in art of two or three dimensional objects. For example in the figure on the right (when the top border is removed and the basic pattern is periodically extended over the whole plane), there is both translational horizontal and vertical symmetry, there is rotation of 90 degrees around the centers of the eightfold rosettes and the intersticial four fold 'decorated crosses' and there are horizontal and vertical reflections through the centres of those rosettes and through the centers of the fourfold crosses. This intuitive idea needs a bit of refinement (see section 2, Classical symmetry, below) but is basically correct and useful. However, in the last 15 years or so, it has become clear that our ideas of symmetry need to be generalized. One cause are the socalled quasicrystals, which, when examined by X-ray crystallography, exhibit five-fold symmetry patterns, which is impossible classically because there are no periodic tilings of three space with five-fold symmetry; another is the case of quantum integrable systems: these derive from classical Liouville integrable systems by quantization, and, provably, the classical symmetry of the classical systems gets destroyed in this process; it turns out that both the symmetry group and the dynamical system need to be deformed together to preserve the symmetry, but the resulting "symmetry" can not anymore be described by a group; instead one needs a certain type of Hopf algebra called a quantum group.

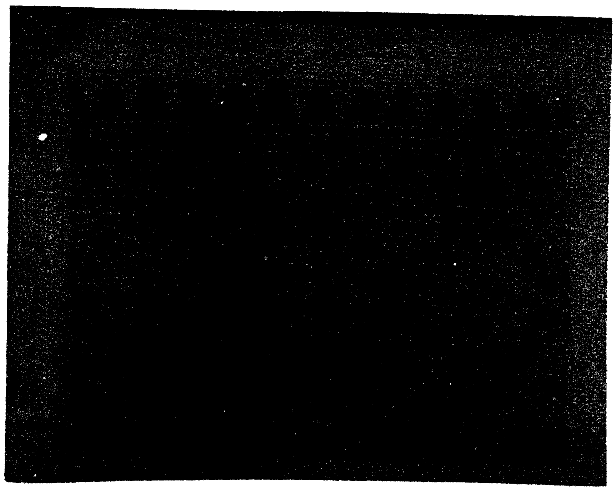

Figure 1

The objects which codify classical symmetry in the sense alluded to above, are groups and Lie algebras (the infinitesimal versions of 'continuous' groups). The applications of groups and Lie algebras are not limited to the cases where there is, so to see, visibly, some symmetry present. There are other, deeper, instances of applications, such as orthogonal polynomials (well understood) and knot invariants (not at all understood) where groups of Lie algebras (or their generalizations) play a decisive role.

2. Classical symmetry.

Let G be a group with unit element $e \in G$ acting on a set X, i.e. there is given a map $\alpha : G \times X \to X$ such that $\alpha(g, \alpha(h, x)) = \alpha(gh, x)$ and $\alpha(e, x) = x$ for all $g, h \in G, x \in X$. Usually one writes simply gx for $\alpha(g, x)$. The *isotropy subgroup* of $x \in X$ is

$$G_x = \{g \in G : gx = x\}$$

and this is the *symmetry* of the object x. This fits nicely which our intuitie ideas of the symmetry of an object in three space or in two space: take for G the appropriate group of Euclidean motions and for X the set of all closed subsets of three space or in two space, as the case may be.

A few more examples follow.

2.1. *Crystallographic symmetry*. Let c be a discrete subset in $\mathbf{R}^3$ together with a labelling from c to some finite set. Think of the elements of c as atoms of various kinds as indicated by their labels. The set c is a classical crystal if it is periodic, i.e. if there is a finite bounded subset of c, which, periodically extended, gives all of c. Let C_{clcr} be the set of all classical crystals. The Euclidean group E_3 acts on C_{clcr} and the isotropy subgroups are the crystal symmetry groups.

2.2. *Automorphisms of algebras*. Take a fixed vector space V. An algebra structure on V is given by a multiplication map (a composition structure) $V \times V \to V$, $(a,b) \mapsto ab$ that is associative: $(ab)c = a(bc)$ and a unit element $1 \in V$: $1a = a = a1$. The group $G = GL(V)$ of vector space autommorphism acts on the set of all algebra structures on V and the isotropy subgroups of a given element A in that set, i.e. an algebra A, is the automorphism group of that algebra

2.3. *Symmetries of differential equations and dynamical systems.* Consider a differential equation (or dynamical system) $\dot{u} = f(u)$ on $\mathbf{R}^n$ (or, more generally, on a differential manifold). A solution is a differentiable map $u : \mathbf{R} \to \mathbf{R}^n$ that satisfies the given differential equation. A symmetry of the differential equation is a diffeomorphism φ of $\mathbf{R}^n$ that takes solutions into solutions: Sometimes one also permits a rescaling of time (dynamical symmetries). These ideas are readily extended to more general differential equations (than flows) and to partial differential equations. It has turned out that is is a good idea to consider more general symmetries in the sense that one not only considers transformations $u \mapsto \varphi(u)$ in which the transformed function $\varphi(u)$ depends only on (the components of) u itself, but generalized transformations in which the result can also be dependent on the derivatives of u up to a certain order (generalized symmetries). Much can be done with these generalized symmetries and many explicit solutions to important equations arise this way, see [11, 26].

One reason (amoung many others) that symmetries are so important is their link with *conservation laws*. Consider a Hamitonian dynamical system on $\mathbf{R}^{2n}$ (or more generally on a symplectic manifold), given by a Hamiltonian function $h : \mathbf{R}^{2n} \to \mathbf{R}$. The evolution in time of a function f on $\mathbf{R}^{2n}$ is given by $\dot{f} = \{h, f\}$ where { , } is the Poisson bracket which in the standard case of $\mathbf{R}^{2n}$ is given by $\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i}$ where the coordinates in $\mathbf{R}^{2n}$ are written $(q_1, \cdots, q_n, p_1, \cdots, p_n)$. Suppose that $\{f, h\} = 0$, i.e. that f is a *conserved quantity*. Then, integrating the flow defined by the function f gives a one parameter group $\varphi(s)$ of symmetries of the original dynamical system. And inversely, a one parameter group of symmetries gives rise to a conserved quantity (*Noether's theorem*).

2.4. *Galois groups*. Let $\overline{\mathbf{Q}}$ be the algebraic closure of the field of rational numbers $\mathbf{Q}$, $G = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ the group of automorphisms of $\overline{\mathbf{Q}}$ and let X be the set of all finite field extensions of the rational numbers. The group G acts on X in the obvious way. If the isotropy subgroup of a finite field extension $F \in X$ is normal in G, $F/\mathbf{Q}$ is a Galois extension and its Galois group is G/G_F. So in this case the symmetry group arises as a quotient. Fields are special algebras so we could also have used the setting of 2.2 above. That gives the same group of symmetries.

3. **Symmetry breaking.**

Consider a square in the plane. As soon as it is slightly deformed, to a rectangle, say, it loses its fourfold rotation symmetry and only the horizontal and vertical reflection symmetries remain. This example and many others concerning symmetries of objects in space give one the feeling that under sufficiently small changes symmetry always becomes less. And indeed there is a theorem to that effect, [21]: Let G be compact Lie group acting smoothly on a compact smooth manifold X, then for every $x \in X$ there is a neighbourhood U such that for every y in U the symmetry G_y of y is smaller than the symmetry G_x of x. Here a subgroup H of G is smaller than a subgroup K of G if there is an element g in G such that $H \subset x^{-1}Kx$. Very little is known more generally. It is universally believed that such a theorem also holds for the crystallographic case (see 2.1 above), but I know of no proof in the literature. Such a symmetry breaking theorem certainly does not hold in full generality. Some examples can be found in [12].

4. **Symmetry and extremality.**

There seems to be a deep relation between symmetry and extremality in the sense that extremal objects tend to have high symmetry. This statement, in its present bald form, needs qualification, see [15], where there is also a selection of examples. One theorem (the *Purkiss principle*) in this direction can be found in [27]: a critical point of a symmetric function for which the induced tangent representation is irreducible is a maximum or a minimum.

5. **Nonclassical symmetry.**

For classical Liouville integrable systems the socalled dressing method provides a large group of symmetries, [29]. However, these dressing transformations do not preserve the symplectic structure on the underlying manifold of the classical Hamiltonion Liouville integrable system, [22]. That means that these symmetries do not survive quantization. It seems that both the dynamical system and the symmetry group must be deformed together to preserve symmetry. The result is a generalized notion of symmetry that is given by a special kind of Hopf algebra called a quantum group. These notion will be described below. Much still needs to be done to understand precisely what happens during the quantization of a classical Liouville integrable systems to a completely integrable quantum system.

There are crystals in nature whose X-ray crystallographic analysis gives five-fold symmetry elements. This is impossible classically. There are also the (related) Penrose tilings of the plane (Penrose uiverses) that have five fold symmetries but are aperiodic. These structures can be understood as projections of higher dimensional classically symmetric objects, [1, 6, 7]. Very possibly the symmetries of these structures can also be understood (but not necessarily better) in terms of a Hopf algebra symmetry.

6. **Group algebra.**

Give a finite group G consider the vector space $\mathbf{C}[G] = \{\sum_{g\in G} a_g g : a_g \in \mathbf{C}\}$ of all (formal) $\mathbf{C}$-linear combinations of the group elements. A multiplication is defined by extending the group multiplication $\mathbf{C}$-linearly. This yields an associative algebra $\mathbf{C}[G]$ with unit e, the *group algebra* of G. Now, to prepare for the idea of a Hopf algebra as carrier of an idea of symmetry observe the following. Given an object M and a symmetry group G of it we have an action of G on M (NB, compared to section 2 above "$M = x$ and $G = G_x$.). Then, on the functions on M we also have an action of G and hence, extending things $\mathbf{C}$-linearly, an action of the group algebra $\mathbf{C}[G]$ on thevector space consisting of the functions on M. This paves the way for the idea of any algebra as a potential symmetry algebra: an algebra A is a symmetry algebra for an object M is there is a nontrivial action of A on the vectorspace Func(M), i.e. if Func(M) is an A-module. This is probably to general to be of much use: there are just to many algebras. For the special class of algebras called Hopf algebras, there are concrete and important examples where they do turn up naturally as symmetry algebras.

Another possible generalization of groups as carriers of symmetry would be to consider semi-groups. This has not yet had much attention.

7. **Tensor product.**

To define the notion of a Hopf algebra we need tensor products. If V is a vector space with basis $\{e_i : i \in I\}$ and W is a vector space with basis $\{f_j : j \in J\}$, then the *tensor product* $V \otimes W$ is the vector space with basis $\{e_i \otimes f_j : i \in I, j \in J\}$. If the dimension of V is m and that of W is n the dimension of the tensor product $V \otimes W$ is mn. There is an obvious natural bijective correspondence between bilinear maps $V \times W \to Z$ into a third vector space Z and linear maps $V \otimes W \to Z$. This can serve as an alternative (and far better) definition of the tensor product which extends to more general cases. If $\alpha : V_1 \to V_2$, $\beta : W_1 \to W_2$ are two linear maps their tensor product is the linear map $\alpha \otimes \beta$ given by $(\alpha \otimes \beta)(e_i \otimes f_j) = \alpha(e_i) \otimes \beta(f_j) = \sum a_i^k b_j^l e_k' f_l'$ if $\alpha(e_i) = \sum a_i^k e_k'$, $\beta(f_j) = \sum b_j^l f_l'$. This corresponds to the *Kronecker product* or *outer product* of matrices.

8. **Hopf algebras**

An *algebra* A over a field k (e.g. $k = \mathbf{C}$, the complex numbers) is a vector space over k with a (bilinear) multiplication (or composition structure) $m : A \times A \to A$, i.e. a linear map $m : A \otimes A \to A$, and a unit element e which is the same as a linear map $e : k \to A$. Associativity is expressed by the relation $m(m \otimes \mathrm{id}) = m(id \otimes m) : A^{\otimes 3} \to A$, and there are also such formulas to express the identity properties.

Dually a *coalgebra* over k is a vector space C with a decomposition structure $\mu : C \to C \otimes C$ and a counit $\varepsilon : C \to k$ such that $(id \otimes \mu)\mu = (\mu \otimes id)\mu$ (coassociativity) and $(\varepsilon \otimes id)\mu = id$, $(id \otimes \varepsilon)\mu = id$ where the natural identifications $k \otimes C \cong C \cong C \otimes k$ are used.

A *bialgebra* $(B, m, e, \mu, \varepsilon)$ over k is a vector space B equipped with both an algebra structure (m, e) and a coalgebra structure (μ, ε) that are compatible in the sense that μ and ε are algebra homomorphisms (or, equivalently, that m and e are coalgebra homomorphisms).

A Hopf algebra H is bialgebra with one more structure element: a linear map $\iota : H \to H$, called *antipode*, such that $m(id \otimes \iota)\mu = id$, $m(\iota \otimes id)\mu = id$.

The group algebra $\mathbf{C}[G]$ of a finite group when equiped with the coalgebra structure, $\mu : g \mapsto g \otimes g$, $\varepsilon(g) = 0$ *for* $g \neq e$, $\varepsilon(e) = 1$, and the antipode $\iota : g \mapsto g^{-1}$ is an example of a Hopf algebra. In this sense Hopf algebras generalize groups.

Two other examples are the algebra of functions on an algebraic groups and the universal enveloping algebra of a Lie algebra. For instance the algebra of algebraic functions on $GL(n, \mathbf{C})$ can be described as the quotient algebra $C[X_1^1, \cdots X_n^n, X] / X \det(X_j^i) = 1$. The comultiplication is given by $X_j^i \mapsto \sum_l X_l^i \otimes X_j^l$, the counit by $\varepsilon(X_j^i) = \delta_j^i$, the Kronecker delta, and the antipode is given by the formula for the matrix inverse. For the universal enveloping algebra $U\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$, the comultiplication is $x \mapsto 1 \otimes x + x \otimes 1$ for $x \in \mathfrak{g}$, and the counit and antipode are given by $\varepsilon(x) = 0$, $\iota(x) = -x$, $x \in \mathfrak{g}$.

9. **Quantum groups**

The notion of a quantum group is not yet completely fixed in the literature; in any case there are two different (but dual) kinds. Roughly a quantum group is a Hopf algebra over the ring of

Laurent series $k((q))$ (or a similar ring such as $k[q,q^{-1}]$) that for q=1 becomes the Hopf algebra of algebraic functions on an algebraic group or the universal enveloping Hopf algebra of a Lie algebra. A comprehensive introduction is [5]. The best known example of the first kind is the quantum group $SL_q(2)$ which, as an algebra, is the quotient of $H_2 = k < t_1^1, t_1^2, t_2^1, t_2^2 >$ by the relations $t_2^1 t_1^1 = q t_1^1 t_2^1$, $t_1^2 t_1^1 = q t_1^1 t_1^2$, $t_2^2 t_2^1 = q t_2^2 t_2^1$, $t_2^2 t_1^2 = q t_2^2 t_1^2$, $t_1^2 t_2^1 = t_2^1 t_1^2$, $t_2^2 t_1^1 - t_1^1 t_2^2 = (q - q^{-1}) t_2^1 t_1^2$, and (the quantum determinant is 1 relation) $t_1^1 t_2^2 - q t_1^2 t_2^1 = 1$. The comultiplication is given by the matrix multiplication formula (see section 11 below). These relations, except the quantum determinant is 1 relation, come from the R-matrix

$$\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q-q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

(or several other possibilities) by the recipe from section 11 below. There are similar quantum groups associated to all the classical simple Lie groups.

10. **R-matrices and the Yang-Baxter equation**

Let V be an n-dimensional vector space, and R an endomorphism of $V \otimes V$; i.e after choosing a basis, R becomes an $n^2 \times n^2$ matrix. For each $i = 1, \cdots, n$ let

$$R_i = \underbrace{Id_V \otimes \cdots \otimes Id_V}_{i-1 \text{ factors}} \otimes R \otimes \underbrace{Id_V \otimes \cdots \otimes Id_V}_{n-i-1 \text{ factors}}$$

With these notations (one form of) the Yang-Baxter Equation (YBE) is $R_1 R_2 R_1 = R_2 R_1 R_2$. These constitute n^6 equations in n^4 variables, so one does not expect, a priori, very many solutions. It is a remarkable fact that there are in fact a good many solutions. The YBE equation originally arose in the work of R J Baxter on solvable models in statistical lattice mechanics, [3] and the work of C N Yang in particle physics theory, [28]. It is no accident that the same equation arose in these two fields.

There are R-matrices for each of the quantum groups associated to the classical simple Lie groups that satisfy the YBE. (More precisely the matrices τR where τ is the switch endomorphism $v \otimes w \mapsto w \otimes v$ of $V \otimes V$ satisfies the YBE in the form given.)

11. **Quantum groups and QIST**

Given any $n^2 \times n^2$ matrix $R = (r_{cd}^{ab})$ one can associate a bialgebra to it as follows. Consider the free associative algebra over k in n^2 indeterminates $H_n = k < t_1^1, \cdots, t_n^n >$. Now consider the expression

(11.1) $\quad RT_1T_2 - T_2T_1R$

where T is the $n \times n$ matrix of indeterminates

$$T = \begin{pmatrix} t_1^1 & \cdots & t_n^1 \\ \vdots & & \vdots \\ t_1^n & \cdots & t_n^n \end{pmatrix}, \quad T_1 = T \otimes Id, \ T_2 = Id \otimes T$$

(so that T_1 and T_2 are $n^2 \times n^2$ matrices). Let $I(R)$ be the ideal in H_n generated by the n^4 elements (11.1). Then the quotient algebra $H_n(R) = H_n / I(R)$ always has a natural bialgebra structure under the standard matrix comultiplication $t_j^i \mapsto \sum t_l^i \otimes t_j^l$.

The expression (11.1) is at the heart of the QIST (Quantum Inverse Scattering Transform) method as developed by the Skt Peterburg (Leningrad) school of Ludwig Faddeev, [9, 10]. Finding solutions of the FCR (Fundamental Commutation Relations) $RT_1T_2 = T_2T_1R$, where T is a matrix of operators, is the same as studying the representations of (the dual of) the bialgebra $H_n(R)$, [13]. Once such a solution has been found, the socalled algebraic Bethe Ansatz provides a systematic approach to find the eigenvalues and eigenvectors of the trace of T. The algebraic Bethe Ansatz itself has interrelations with the representation theory of the symmetric groups (Young diagrams and such), [16, 17], a matter that asks for further investigation.

12. **Quantum groups and *q*-special functions**

It is one of the remarkable discoveries of the mid 20-th century (by Eugene Wigner, N Ya Vilenkin, Willard Miller Jr)that the special functions of mathematical physics, such as Jacobi polynomials and other orthogonal polynomials, and Bessel functions, basically arise as matrix coefficients of representations of Lie groups, see [25] for a comprehensive survey. In the mid 19-th century, long befor quantum theory, it was discovered that it is possible to insert a parameter q in the various known families of special functions in such a way that many of their properties (such as orthogonality) were preserved and such that for q=1 the original functions reappeared. It is a quite recent discovery, simultaneously by several (groups of) people, amoung which Koornwinder, see [18, 19, 20], and [25], that these q-special functions arise from quantum groups in much the same way, i.e. as coefficients of representations of quantum groups. See also [8] for narrowly related work. It is a historical accident that the letter q of q-special functions, adopted in the 1850-ies, is also the q of 'quantum'.

13. **Knots and links and their invariants**

A knot is the image of a piecwise-differentiable diffeomorphism of the cicle into three space; a link is the disjoint union of several knots (which maybe linked, whence the name). Above are some examples.

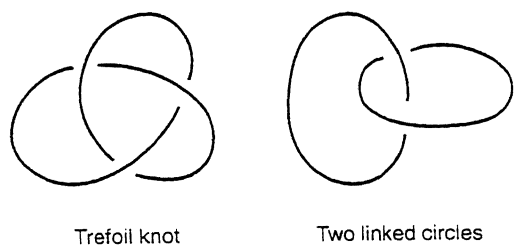

Figure 2

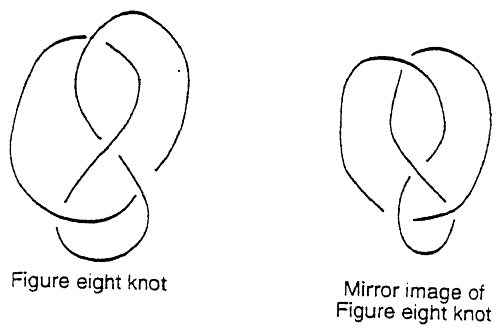

Figure 3

Two knots or links are considered equivalent, if one can be changed continuously to the other without cutting and gluing (ambient isotopy). This is precisely the intuitive idea of the equivalence of two knots. For instance the knot in figure 4 is equivalent to a simple circle in tree space (the unknot). The figure eight knot and its mirror image (see figure 3 above) are equivlent, but the mirror image of the trefoil knot (see figure 2 above) is not equivalent to the trefoil itself. The example below, figure 5, due to Tietze, shows two knots: one of these is trivial, the other not. They differ in only one spot where an overcrossing has been changed to an undercrossing.

Examples like these show the need for algorithmically computable objects (knot and link invariants), which are the same for equivalent links and which, ideally, distinguish between different ones. Whether this ideal can be realized is still an open question. However, in the last few years a number of most striking new invariants have been discovered, notably the so-called Jones and Kaufmann polynomials. Rather mysteriously—there is no Lie algebra of symmetry in sight—they come from quantum groups and solutions of the Yang-Baxter equations.

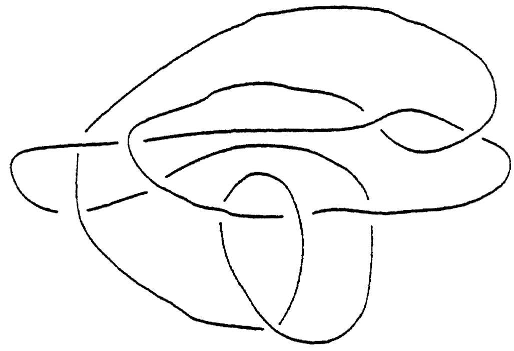

Figure 4

14. **Vassiliev invariants**

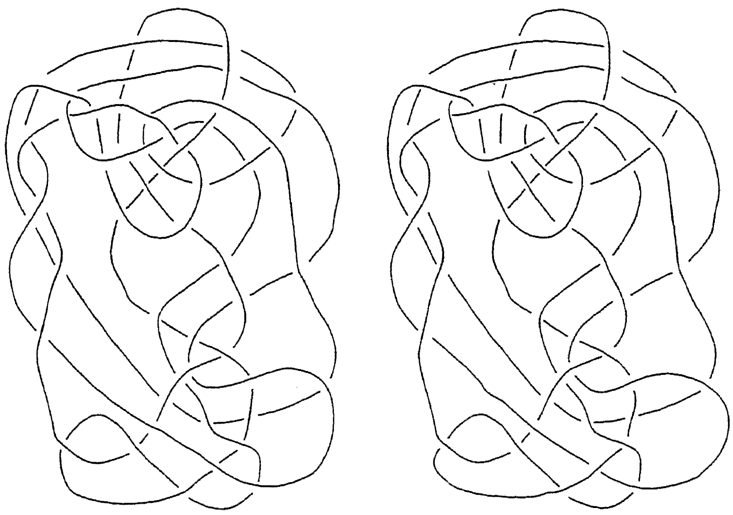

Figure 5

For a singular link a finite number of transversal self intersections are allowed; i.e. instead of inbeddings we now consider immersions. There is an analogous idea of equivalence, called rigid vertex equivalence (which is a bit stronger than the intuitive idea of changing a singular link continuously to another). Each oriented knot invariant V can be extended inductively to an invariant $V^{(m)}$ for singular knots with precisely m self intersections by the formulas $V^{(0)} = V$ and

$$V^{(m)}(\times) = V^{(m-1)}(\times) - V^{(m-1)}(\times)$$

A knot invariant V is a Vassiliev invariant of order m if $V^{(m)}$ is identicaly zero. There are many Vassiliev invariants. Following [4], let us call the invariants arising from quantum groups by Turaev's formula, see section 17 below, *quantum group invariants*. Now, loc. cit., replace q by e^x and expand in powers of x, then the coefficient of x^m is a Vassiliev invariant of order m.

All known Vassiliev invariants come from Lie algebras, and, conjecturally, all do, [2], a matter that is more than a bit mysterious.

15. **Braid group**

A braid on m strings (strands) is formed as follows. Take m points on a line in an upper horizontal plane and m points vertically below them in a lower horizontal plane. Attach strands connecting each upper point to some lower point in such a way that each horizontal plane between the upper and lower one (including those two themselves) intersects the stands in precisely m points. Figure 6 shows an example of a braid on five strands.

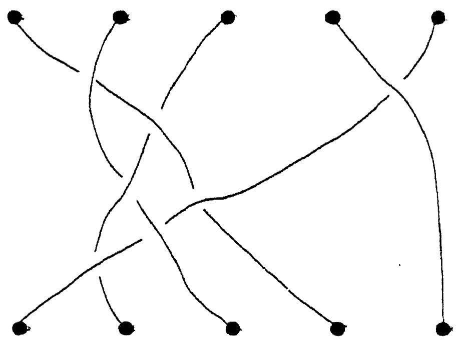

Figure 6

There is a natural way to compose braids: identify the lower horizontal plane of one braid with the upper one of another on the same number of strands. This gives a new braid. Figure 7 shows the composition of two braids on two strands.

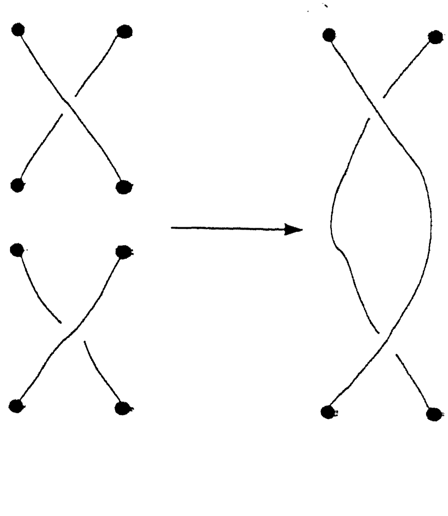

Figure 7

Two braids are equivalent if they can be deformed into each other by an ambient isotopy keeping the end points fixed and without the strands passing out of the area bounded by the upper and lower horizontal planes. Non-uniform (and uniform) stretching and shrinking is also allowed, i.e. moving the two bounding planes up and down and/or moving the endpoints on the upper and lower bounding planes closer together or further apart. The trivial braid on m strands consists (of the equivalence class) of m strands dropping straight down. For instance the two braids in figure 8 are equivalent; the right braid is the trivial one on two strands. It readily follows that the equivalence classes of braids on m strands form a group. It is called the *Artin braid group* B_m. Now consider the elementary braids on m strands σ_i, $i = 1,\cdots,m-1$, where σ_i has the i-th strand undercrossing the $(i+1)$-th one, and nothing else, see figure 9 below. It is now also easy to see that each braid on m strands is equivalent to one composed of the elementary braids $\sigma_1,\cdots,\sigma_{m-1},\sigma_1^{-1},\cdots,\sigma_{m-1}^{-1}$ where the inverse elementary braids are exactly like the ones in figure 9 except that the i-th strand overcrosses number $i+1$. A much deeper result is *Artin's theorem*: the braid group on m strands is generated (as a group) by the elementary braids σ_i, $i=1,\cdots,m-1$, and the relations are engendered by

$$\sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1},\ i=1,\cdots,m-2$$
$$\sigma_i\sigma_j=\sigma_j\sigma_i,\ \text{for } |i-j|\geq 2$$

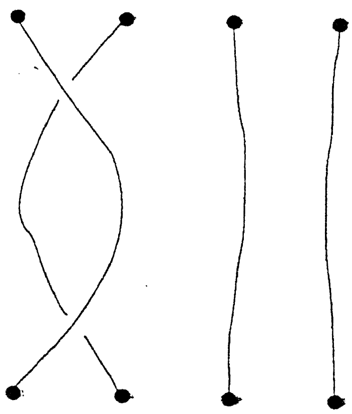

Figure 8

16 Links and braids

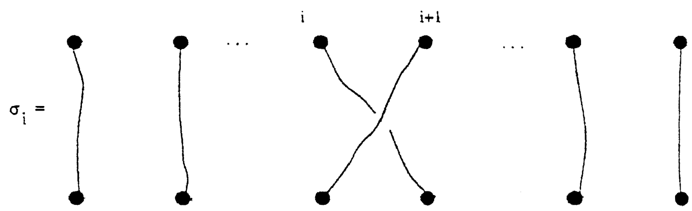

Figure 9

Given a braid it is easy to obtain a knot from it: simply connect the upper points to the lower ones by nonintersecting and noninterlacing piece-wise smooth curves. It does not matter how this is done. The resulting links will be ambient isotopic. Figure 10 illustrates the process. Every link (equivalence class) can be obtained this way (Alexander's theorem) and it is also precisely known when two braids give the same link (Markov's theorem). As a matter of fact the closure depicted in figure 10 is the trefoil knot as can be seen by inspecting figure 11.

17. Quantum group link invariants

An extended Yang-Baxter operator is a quadruple (R,v,α,β) consisting of an invertible $n^2\times n^2$ matrix R satisfying the YBE (see section 10 above), an $n\times n$ matrix v, and two scalars α,β. In addition to the YBE for R these data are required to satisfy

$$(v\otimes v)R=R(v\otimes v),\ \ Tr_2(R^{\pm1}(v\otimes v))=\alpha^{\pm1}\beta v$$

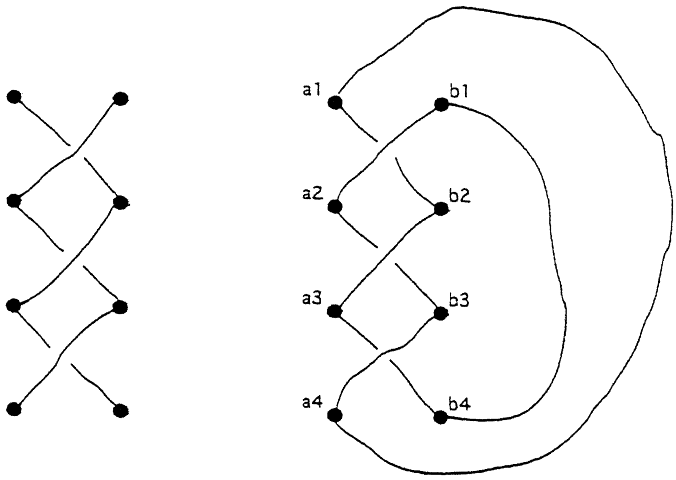

Figure 10

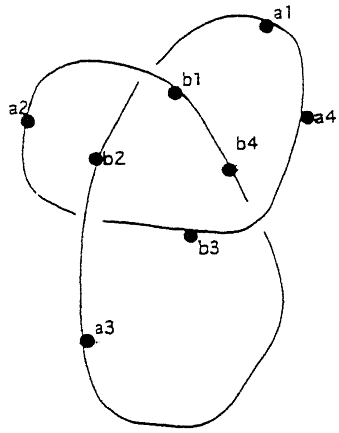

Figure 11

Here, if $M = (m_{cd}^{ab})$ is an $n^2 \times n^2$ matrix (with the usual lexicographic ordening of the upper and lower indices), then $Tr_2(M) = N = (n_j^i)$ is the $n \times n$ matrix with $n_j^i = m_{j1}^{i1} + \cdots + m_{jn}^{in}$. Given an extended Yang-Baxter operator, *Turaev's formula*, [23]

$$T_R(\xi) = \alpha^{-w(\xi)}\beta^{-m}Tr(\rho_R(\xi)v^{\otimes m})$$

defines an oriented link invariant. Here $\xi = \sigma_{i_1}^{\varepsilon_1} \cdots \sigma_{i_r}^{\varepsilon_r}$ is an element of the braid group B_m on m strands, $w(\xi) = \varepsilon_1 + \cdots + \varepsilon_r$, and ρ_R is the representation of the braid group defined by the invertivle YBE solution R which sends the generator σ_i to R_i (see section 10 above). The Jones polynomial and the Kaufmann polynomial can be obtained from these Turaev quantum group invariants when one takes for the R-matrices those that define the one parameter quantum groups associated to the classical simple Lie algebras.

18. **Multiparameter R-matrices**

The R-matrices that define the A-type quantum groups $GL_q(n)$ satisfy the condition

(18.1) $\quad r_{cd}^{ab} = 0, \text{ unless } \{c,d\} = \{a,b\}$

In [14] all invertible solutions are determined of the YBE that satisfy this additional condition (18.1) These solutions have a block structure with each block in turn consisting of a number of interconnected cells. The blocks are also interconnected in various non-trivial ways. The single block solutions with size 1 cells give precisely the $\binom{n}{2} + 1$ parameter A-type quantum groups that have been independently discovered recently (1990/1991) by at least six (groups) of authors. These extend to extended Yang-Baxter operators, but do not define new invariants (the extra parameters drop out). Multiple block solutions extend to extended YB operators if and only if a certain number attached to each block is the same. This does potentially give new invariants, [14]. It remains to determine how strong these invariants are and if they give more than is already available. The B-, C-, and D-type quantum group R-matrices satsify a similar but more complicated condition (18.1). The solutions satisfying that condition can also be largely determined, [24].

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